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2-radical subgroups of the Conway simple group Co_1

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1 Introduction

Let G be a finite group and p be an element of $\pi(G)=\{p$: prime $|$ p divides $|G|\}$. Put $\tilde{\mathcal{B}}_p(G)=\{U$: p-subgroup $\subseteq G \mid O_p(N_G(U))=U\}$ and $\mathcal{B}_p(G)=\tilde{\mathcal{B}}_p(G)-\{1\}$. An element of $\mathcal{B}_p(G)$ is called a p-radical subgroup of G. $\mathcal{B}_p(G)$ plays an important role in the various fields. For example, $\Delta(\mathcal{B}_p(G))$ gives us a valuable information when we verify the Dade's conjecture for G. Here $\Delta(\mathcal{B}_p(G))$ is a simplicial complex whose vertex set is $\mathcal{B}_p(G)$, and its simplex is each chain of elements of $\mathcal{B}_p(G)$ with respect to natural inclusion in $\mathcal{B}_p(G)$. $\Delta(\mathcal{B}_p(G))$ is called the p-radical complex of G. Furthermore it is known that the alternating-sum decomposition of mod p cohomology of G is

$$\tilde{H}^n(G,\mathbf{Z}_p)=\sum_{\sigma\in\Delta(\mathcal{B}_p(G))/G}(-1)^{\dim(\sigma)}\tilde{H}^n(G_\sigma,\mathbf{Z}_p),$$

where n is any non-negative integer, G_σ is the stabilizer of a simplex σ, and $\Delta(\mathcal{B}_p(G))/G$ is a set of the representatives of G-orbits of $\Delta(\mathcal{B}_p(G))$ (See [5]). Hence the calculation of a group cohomology reduces to the calculation of smaller groups. On the other hand, $\Delta(\mathcal{B}_p(G))$ can be regarded as a geometry for G. Recently, for a sporadic simple groups G, $\Delta(\mathcal{B}_p(G))$ is investigated in this direction very much, and it is closely connected with the essential p-local geometry for G. $\Delta(\mathcal{B}_p(G))$ is determined by S. D. Smith, S. Yoshiara and et al. for some sporadic simple groups G and $p\in\pi(G)$. The purpose of this note is to announce [3], namely determination of $\mathcal{B}_2(Co_1)$ up to conjugacy, where Co_1 is the Conway simple group.

2 Known and new results about p-radical subgroups

The following lemma is one of the most basic results on p-radical subgroups.

Lemma 1 ([4; Lemma1.10]) *Let G be a finite group and $p\in\pi(G)$. If $U\in\mathcal{B}_p(G)$ with $N_G(U)\subseteq M$, where M is a subgroup of G, then $O_p(M)\subseteq U$. In particular, If $O_p(M)\neq U$ then $U/O_p(M)\in\mathcal{B}_p(M/O_p(M))$.*

Lemma 1 implies that we can find p-radical subgroups inductively.

Corollary 1 *Let G be a finite simple group, M be a maximal subgroup of G and $p \in \pi(M)$. If $O_p(M) \neq 1$ then $\mathcal{B}_p(M) = \{O_p(M), U \mid U/O_p(M) \in \mathcal{B}_p(M/O_p(M))\}$.*

Theorem 1 ([1]) *Let G be a group of Lie type over a field of characteristic p. Then $\mathcal{B}_p(G) = \{O_p(U) \mid G \supseteq U = \text{parabolic subgroup}\}$.*

Proposition 1 *For H and K are finite groups and $p \in \pi(H \times K)$, $\widetilde{\mathcal{B}}_p(H \times K) = \{V \times K \mid V \in \widetilde{\mathcal{B}}_p(H),\ W \in \widetilde{\mathcal{B}}_p(K)\}$ holds.*

Proposition 2 *Let A be a finite group with a normal subgroup G of a prime index p. Then for any $U \in \mathcal{B}_p(A)$, $U \cap G = \{1\}$ or $U \cap G \in \mathcal{B}_p(G)$.*

In this case we have $\{U \in \mathcal{B}_p(A) \mid U \subseteq G\} \subseteq \mathcal{B}_p(G)$. On the other hand, for $U \in \mathcal{B}_p(A)$ with $U \not\subseteq G$, there exists an element $x \in G$ such that $U = (U \cap G)\langle x \rangle$. We can easily see that $U_1 = U \cap G \in \widetilde{\mathcal{B}}_p(G)$ and $|U : U_1| = p$. Hence it suffices to determine $\mathcal{B}_p(G)$ essentially.

Proposition 3 *Let G be a finite group of Lie type over a field of characteristic p, and σ be a field automorphism of G of order p. Then $\{U \in \mathcal{B}_p(G\langle\sigma\rangle) \mid U \subseteq G\} = \mathcal{B}_p(G)$.*

3 Application

We consider the case $G = Co_1$ and $p = 2$. Let (Λ, q) be the Leech lattice, that is, (Λ, q) is the 24-dimensional even unimodular lattice which has no vector $\mathbf{v}$ with $q(\mathbf{v}) = 2$. Let $\mathrm{Aut}(\Lambda, q) := \{\sigma \in O(\mathbf{R}^{24}, q) \mid \Lambda^\sigma = \Lambda\}$. $\mathrm{Aut}(\Lambda, q)$ is called the Conway group, which will be denoted $\cdot 0$. Its center $Z = Z(\cdot 0)$ is of order 2, and the factor group $Co_1 := \cdot 0/Z$ is a simple group, which is also called the Conway group. The following remark is straightforward from our definitions

Remark 1 *Let G be a finite group and $p \in \pi(G)$. If $U \in \mathcal{B}_p(G)$ with $N_G(U) \subseteq M$, where M is a subgroup of G, then $U \in \mathcal{B}_p(M)$.*

The local subgroups of Co_1 have been classified by Curtis [2].

Theorem 2 ([2; Theorem 2.1]) *For any elementary abelian 2-subgroup E of $\cdot 0$, $N_{\cdot 0}(E)/Z$ is contained in a conjugate of one of the following seven groups.*

$$\begin{array}{lll} L_1 = 2_+^{1+8} \cdot \Omega_8^+(2) & L_4 = 2^{11}{:}\, M_{24} & L_7 = (A_6 \times PSU_3(3)){:}\,2 \\ L_2 = 2^{4+12} \cdot (S_3 \times 3Sp_4(2)) & L_5 = Co_2 & \\ L_3 = 2^{2+12}{:}(S_3 \times L_4(2)) & L_6 = (A_4 \times G_2(4)){:}\,2 & \end{array}$$

Remark 1 and Theorem 2 imply $\mathcal{B}_2(Co_1) \subseteq \{U^g \mid g \in Co_1,\ U \in \mathcal{B}_2(L_i)\ (1 \le i \le 7)\}$. We can determine $\mathcal{B}_2(L_i)$ systematically by using the results in the previous section as follows.

$\mathcal{B}_2(L_i)$ $(1 \le i \le 5)$: It suffices to determine 2-radical subgroups of $\Omega_8^+(2)$, S_3, $3Sp_4(2)$, $L_4(2)$, M_{24} and Co_2 by Corollary 1 and Proposition 1. We can find them from [4], [6] and Theorem 1.

$\mathcal{B}_2(L_i)$ $(i = 6, 7)$: Essentially it suffices to determine 2-radical subgroups of A_4, A_6 $G_2(4)$ and $PSU_3(3)$ by Propositions 1, 2 and 3. The cases A_4 and A_6 are straightforward. We can easily determine $\mathcal{B}_2(G_2(4))$ and $\mathcal{B}_2(PSU_3(3))$ by Theorem 1.

Now we find the candidates for $\mathcal{B}_2(G)$, that is, we find $\mathcal{B}_2(L_i)$ $(1 \leq i \leq 7)$. Next we have to examine which element of $\mathcal{B}_2(L_i)$ actually belongs to $\mathcal{B}_2(G)$ for each i $(1 \leq i \leq 7)$. However when we examine we need detailed arguments. Then we have the following result.

$\mathcal{B}_2(Co_1)$ consists of exactly 30 classes, and the representatives and the normalizers of them in Co_1 are as shown in TABLE 1, where $\{P_i\}_{1 \leq i \leq 15}$ and $\{N_i\}_{1 \leq i \leq 7}$ are the sets of representatives of $\mathcal{B}_2(O_8^+(2))$ and $\mathcal{B}_2(L_4(2))$ respectively.

Table 1: $\mathcal{B}_2(Co_1)$

representative T	$N_{Co_1}(T)$
$R = 2_+^{1+8}$	$R^{\cdot} O_8^+(2)$
$R.P_i$ $(1 \leq i \leq 15)$	$R.N_{O_8^+(2)}(P_i)$
$E = 2^{11}$	$E : M_{24}$
$Q = 2^{4+12}$	$Q^{\cdot}(S_3 \times 3S_6)$
$Q : S = 2^{4+12} : 2$	$Q^{\cdot}(S \times 3S_6)$
$Q_1 = 2^{2+12}$	$Q_1 : (S_3 \times L_4(2))$
$Q_1 : N_i$ $(1 \leq i \leq 7)$	$Q_1 : (S_3 \times N_{L_4(2)}(N_i))$
$V = 2^2$	$(A_4 \times G_2(4)) : 2$
$V : \langle\sigma\rangle = 2^2 : 2$	$(V \times G_2(2)) : \langle\sigma\rangle$
$F = 2^2$	$(S_4 \times PSUU_3(3)) : 2$

Remark. Let G be a finite group and $p \in \pi(G)$. A p-subgroup chain $C : P_0 < P_1 < \cdots < P_n$ is called a radical p-chain of G if it satisfies $P_0 = O_p(G)$ and $P_i = O_p(\cap_{j=0}^{i} N_G(P_j))$ for all i. We can easily determine all the radical 2-chains of Co_1 up to conjugacy by using Theorem 1, Proposition 1, [6] and the main result of this note.

References

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